Classifying amenable operator algebras

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1. Introduction

Goal: survey some classification results for amenable operator algebras

40s . . . 70s . . . present

Operator algebras

Example: $\mathcal{B}(\mathcal{H})$, cts. linear operators on $\mathcal{H}$

Hilbert space

- algebraic structure: $*$-algebra, $\langle T^*\xi, \eta \rangle = \langle \xi, T\eta \rangle$
- analytic structure: $\|T\| = \sup\{\|T\xi\|_{\mathcal{H}} : \|\xi\| \leq 1\}$, Banach space
- e.g. $\mathcal{H} = \mathbb{C}^n \rightsquigarrow$ $n \times n$ matrices, $M_n(\mathbb{C})$

$\star$ C^*-algebras

- $A \subseteq \mathcal{B}(\mathcal{H})$, closed in $\|\cdot\|$-topology
- unital commutative case: $C(X)$

topological flavor

von Neumann algebras

- $M \subseteq \mathcal{B}(\mathcal{H})$, closed under pt-wise limits
- commutative case: $L^\infty(X, \mu)$

measure theoretic flavor

Ex. group algebras

- Γ : countable discrete grp. $\rightsquigarrow$ Hilbert sp. $\ell^2(\Gamma)$ (square summable $f : \Gamma \to \mathbb{C}$)
- left regular rep : $\gamma \mapsto \lambda_\gamma \in B(\ell^2(\Gamma))$ (δ_γ) , $\lambda_\gamma(f) = f(\gamma^{-1}\cdot)$ $\lambda_\gamma \delta_{\gamma'} = \delta_{\gamma\gamma'}$
- $C^*_\lambda(\Gamma)$ = $\|\cdot\|$ - closure of $*$-subalg. generated by $\{\lambda_\gamma : \gamma \in \Gamma\}$

  $vN(\Gamma)$ = similar, using pt.-wise lim's.

Generalizes Fourier transform: Γ abelian $\Rightarrow$ $C^*_\lambda(\Gamma) = C(\hat{\Gamma})$ dual of Γ

E.g. $C^*_\lambda(\mathbb{Z}) = C(\mathbb{T}) \not\cong C(\mathbb{T}^2) = C^*_\lambda(\mathbb{Z}^2)$

$vN(\mathbb{Z}) = L^\infty(\mathbb{T}) \cong L^\infty(\mathbb{T}^2) = vN(\mathbb{Z}^2)$

Ex. Dynamics $\alpha : \Gamma \to \mathrm{Homeo}(X)$

- action $\Gamma \overset{\alpha}{\curvearrowright} X$ $\rightsquigarrow$ induced action $\Gamma \curvearrowright C(X)$, $\gamma f = f \circ \alpha_\gamma^{-1}$

- combine $C(X)$, $C^*_\lambda(\Gamma)$ — similar to semi-direct product $N \rtimes H$ of groups

$\rightsquigarrow$ crossed product $C(X) \rtimes_\lambda \Gamma$

- When X/α is "reasonable" (e.g. α free, proper),

$C(X/\alpha)$ and $C(X) \rtimes_\lambda \Gamma$ have the "same" representation theory

Famous example: irrational rotation algebra A_θ

- $\theta \in \mathbb{R} \setminus \mathbb{Q}$. $\mathbb{Z} \curvearrowright \mathbb{T}$ by $e^{2\pi i t} \longmapsto e^{2\pi i (t+\theta)}$

- $A_\theta := C(\mathbb{T}) \rtimes_\lambda \mathbb{Z}$

2. Factors, finite-dimensional approximations, amenability.

Factor: a vN alg. with no vN alg. ideals ($\Leftrightarrow$ has trivial center)

Ex.

- $\overline{\bigotimes_{1}^{\infty} M_2(\mathbb{C})}^{SOT}$ ← normalized matrix traces
- $vN(\Gamma)$, Γ infinite and icc (∞ conjug. classes)
- $L^\infty(X) \rtimes \Gamma$, $\Gamma \curvearrowright X$: free, ergodic, probability-measure preserving action

$\Bigg\}$

- all of these have a trace: cts. linear functional τ with $\tau(xy) = \tau(yx)$
- ∞-dim'l factors with a trace are called II_1 factors.

Early classification results.

$\bigcup F_n$ dense

Def. a vN alg. is *hyperfinite* if it can be approximated internally by finite dimensional alg's

$\bigoplus M_{n_i}(\mathbb{C})$

Ex.

- $\overline{\bigotimes_1^\infty M_2(\mathbb{C})}^{SOT} \cong \overline{\bigotimes_1^\infty M_3(\mathbb{C})}^{SOT} \cong vN(\Gamma)$, Γ = finite permutations on $\mathbb{N}$, or any locally finite icc Γ

'43 **Theorem** (Murray – von Neumann) $\exists!$ hyperfinite II_1 factor, $\mathcal{R}$.

Glimm (1960): not so for C^*-alg's! $\overline{\bigotimes_1^\infty M_2(\mathbb{C})}^{\|\cdot\|} \not\cong \overline{\bigotimes_1^\infty M_3(\mathbb{C})}^{\|\cdot\|}$.

Classified a C^*-analog of hyperfinite II_1 factors.

Amenability

Γ (discrete group) is *amenable* if $\exists$ finitely additive left-invariant probability measure on its subsets (a "mean").

Ex.
- finite grps, abelian grps.
- closed under subgroups, extensions, quotients, direct limits
- non-ex.: free groups $\mathbb{F}_n$ $(n \geq 2)$ — see: Banach–Tarski

Can define analog for C^*- and vN alg's. Have: $C^*_\lambda \Gamma$ amenable $\Leftrightarrow$ $vN(\Gamma)$ amenable $\Leftrightarrow$ Γ amenable

Definition is abstract — as opposed to concrete def. of hyperfiniteness.

Breakthrough result:

Theorem (Connes '76): a vN alg. is hyperfinite iff it is amenable.

(In particular, every amenable II_1 ~~factor~~ is $\cong \mathcal{R}$.)

- E.g.: $L^\infty(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ is independent of $\theta \in \mathbb{R} \setminus \mathbb{Q}$
- Led to further breakthroughs, e.g.: all free ergodic prob. measure preserving actions of infinite amenable grps are orbit equivalent.
- a major ingredient: an amenable II_1 factor M satisfies $M \cong M \bar{\otimes} \mathcal{R}$.

McDuff

Classifying approx. fin. dim'l C^*-algebras.

- internal approximations by fin. dim'l alg's in norm: AF algebras

Theorem (Elliott '77) AF algebras are classified by their K_0 groups.

Op. alg. K-theory: NC extension of Atiyah-Hirzebruch's top K-theory

Distinction: $L^\infty([0,1])$ — hyperfinite vNa ; $C([0,1])$ — not AF

3. More recent progress

- generalizations? inspiration from vNa side
- Elliott: classifies large class of AT alg's — lim's of $\bigoplus M_n(C(\mathbb{T}))$ — conjectures classification holds very generally (simple, amenable, ...)
- positive results for more general types of lim's; includes alg's not obviously of that form — e.g. A_θ

- also challenges: Rørdam, Toms produced counterexamples; not possible to fix by modifying invariant (reasonably)
- Toms-Winter regularity: study properties that make a C^*-alg. "well behaved"
- analogy with vNa factors, where natural notions of amenability are equiv.

amenable — hyperfinite — McDuff — tracial comparison of proj's

The Toms-Winter ~~conjecture~~ essentially a thm.:

- A: unital, sep., simple, amenable, $\neq M_n(\mathbb{C})$

TFAE:

(i) A has finite $\boxed{\text{nuclear dimension}}$

(ii) A absorbs the Jiang-Su alg. $\mathcal{Z}$ tensorially: $\boxed{A \otimes \mathcal{Z} \cong A}$

(iii) A has strict comparison of positive elements

- NC analog of covering dim:

  $\dim_{nuc} C(X) = \dim X$
- connections to dim theories for groups, dyn. systems ...

- analog of McDuff: $M \otimes \mathcal{R} \cong M$
- ∞-dim'l analog of $\mathbb{C}$; K-theory can't tell them apart

"regular": satisfies one (all) of these conditions

The Classification Theorem.

Along with J. Gabe, C. Schafhauser, A. Tikuisis, and S. White, we have obtained a proof of the following:

$$\boxed{\text{Simple, separable, amenable, regular } C^*\text{-alg's in the UCT class are classified by K-theory and traces.}}$$

Ex. Thm. applies to $C(X) \rtimes_\lambda \Gamma$ if e.g.

- X is compact metric space of finite covering dimension;
- Γ is locally of subexponential growth;
- action is free & minimal.

More concrete ex: $\mathbb{Z} \curvearrowright \mathbb{Z}_2 = \varprojlim \mathbb{Z}/2^n\mathbb{Z} \subseteq \prod \mathbb{Z}/2^n\mathbb{Z}$

$\hookrightarrow \mathbb{Z}[\frac{1}{2}], \mathbb{Z}$