

# Classifying amenable operator algebras

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TCV

# 1. Introduction

Goal: survey some classification results for amenable operator algebras

present

40s

70s

### INTRODUCTION

1. The object of this paper is the investigation of the set of operator rings which are factors and which are the earlier publications of the authors. (Cf. [3], [9] and [8].) It begins also for the case of factors of which there, below—cf. [3], p. 138, Definition 3.1.2, p. 172, Theorem 3.1.1—case (I)—have been exhaustively dealt with p. 173, Lemma 3.6.1, p. 189, Definition 3.2.1 and 3.2.2, 3.2.3.) The partly infinite case—i.e., the case (II)—is still open and we have, at least for the time being, merely seen to (3) deals mainly with these factors. Thus we see that (3) case—i.e., the case (II)—and they are our main objective. An added justification of this program may be found in all factors those of the finite continuous case—case (II), immediate interest. (Cf. [3], part V, [9], Chapter IV and A.) It will be seen however that the discrete case can be treated with nearly any extra effort. So we shall deal not only with the purely infinite case.

For the discrete and continuous cases, the finite case (the cases (I) and (II)) respectively—can be adequately help. (Cf. Theorem IX and Lemma 3.1.6.) Therefore we refer to the finite case. And since the discrete case—(I, 2, ...)—is just the finite order matrix rings, this is abbreviated continuous finite case (II).

2. Let us now state the main problems of isomorphism. Consider an operator ring  $M$  in a Hilbert space  $\mathfrak{H}$  which is not central  $M$  in any other way. For the significance of 288-289, Definition 1-3. Then there exist two kinds of those which can be expressed in terms of the entity 1 (the  $\mathfrak{H}$  operators all in any complete  $\mathfrak{H}$ ),  $\mathfrak{H} \oplus \mathfrak{H}$ ,  $\mathfrak{H}$  if only to the operators belonging to  $M$ . Second those which, with, e.g. operators outside of  $M$ , elements of  $\mathfrak{H}$ , etc. The preceding, the latter case are not, we shall also deal. These notions were already investigated by the second as it seems worth while to formulate this distinction in the text. Let, for each  $i = 1, 2$ , a Hilbert space  $\mathfrak{H}_i$  and an operator  $T_i$

Introduction. In this paper we study  $C^*$ -algebra closure of strictly ascending sequences of full  $C^*$ -algebras uniformly hyperfinite. The weak closure of such a sequence were first proved that all such factors are isomorphic,  $\mathfrak{H}$  all isomorphic. In [1] we classify uniformly hyperfinite algebras type (I, II) and obtain a characterization: we identify the pure states and the pure states of  $\mathfrak{H}$ . The  $w^*$ -closure of the pure states of  $\mathfrak{H}$  of all states of the algebra. This is not the first set of pure states is not closed, cf. [1]. In [3] we mention of uniformly hyperfinite algebras acc in [4] and 5 we study certain representations.

The author is pleased to record his gratified for many helpful suggestions, for simplification of the present version of the research in this paper, dissertation at Columbia University.

We assume all algebras have a unit  $\{e_{ij} \mid i, j = 1, \dots, n\}$  of operators on a Hilbert space  $\mathfrak{H}$  (the identity operator on  $\mathfrak{H}$ ), and if  $e_{ij} = e_{ij}^2 = e_{ij}$  for these matrix units, we say  $\{e_{ij}\}$  is a compact. The extreme points are called pure linear subspace of a  $C^*$ -algebra  $\mathfrak{K}$  and if  $\mathfrak{K}$  is positive normalized linear functional. The (non trivial) states of  $\mathfrak{K}$ . If  $\tau$  is a state of  $\mathfrak{K}$  then  $\tau$  on a Hilbert space  $\mathfrak{H}$ , and an  $\mathfrak{H}$   $\mathfrak{H}$ , with  $\tau$  is pure if and only if  $\mathfrak{H}$  is irreducible (see [6] depending upon  $\tau$  and perhaps other variables, designate the function  $\tau \rightarrow \mathfrak{H}$ ).

### Classification of injective

Cases II, III, III<sub>1</sub>, and III<sub>2</sub>

By A. CONNES

### Introduction

A von Neumann algebra  $M$ , acting in a Hilbert space  $\mathfrak{H}$ , is a subspace of the Banach space  $\mathfrak{B}(\mathfrak{H})$  in  $X$ ,  $M$  is the range of a projection of norm one on  $\mathfrak{B}(\mathfrak{H})$ .

THEOREM 1. All injective factors of type I Hilbert spaces, are isomorphic.

We now mention several applications, all of which remained open for a long time.

CONSEQUENCE 2. All subfactors of the Murray-von Neumann factor  $R$  [20] are isomorphic to  $R$  or  $R \otimes R$  [4, 4.57].

This shows that  $R$  can be characterized as the unique factor, it can be imbedded in all infinite type I factors, the only one. Also all von Neumann algebras is a product of von Neumann algebras and a von Neumann algebra  $A \otimes R$ , where  $A$  is a von Neumann algebra. Thus all von Neumann subalgebras of  $R$  are isomorphic to the product of  $R$  and a von Neumann algebra.

Another remarkable property of the factor  $R$  is that it is the unique factor, it is characterized by a bi-invariance property, and it is a type I factor.

### On the Classification of Inductive Limits of Semisimple Finite-Dimensional

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### STRUCTURE OF NUCLEAR $C^*$ -ALGEBRAS QUASIDIAGONALITY TO CLASSIFY AGAIN

Wielhelm Winter

### Abstract

I give an overview of recent developments in the theory of separable, simple, nuclear  $C^*$ -algebras. I will discuss quasidiagonality and amenability for classification, and its interplay with internal and external approximations.

### Introduction

A  $C^*$ -algebra is a (complex) Banach- $*$  algebra such that the norm is compatible with the  $*$ -operation. Equivalently,  $C^*$ -algebras may be thought of as norm-closed operators on Hilbert spaces. A von Neumann algebra is to the weak operator topology. Examples of  $C^*$ -algebras compact Hausdorff spaces, section algebras of vector bundles, constant maps of group algebras. Group  $C^*$ -algebras in particular there is a full one, which is universal with respect to the group, and a reduced one, which is the norm completion. Similar constructions can be associated with topological groups and their crossed product construction.

From the 1970s on it became clear that the notion of amenability for groups, with its many equivalent formulations, can be rephrased, in almost as many ways, for operator algebras as well. Some of these notions are more or less directly carried over from group

### CLASSIFYING \*-HOMOMORPHISMS I: SIMPLE NUCLEAR $C^*$ -ALGEBRAS

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### INTRODUCTION

The landscape of operator algebras was revolutionized in the 1970's through Connes' structural theorem for amenable von Neumann algebras, leading to their complete classification [20, 56]. Operator algebras come in two main types: von Neumann algebras and  $C^*$ -algebras, which have the flavor of measure and topology, respectively. Indeed, the Gelfand transform identifies abelian von Neumann and  $C^*$ -algebras with functions on measurable and locally compact Hausdorff spaces, respectively.

# Operator algebras

Hilbert space  
↙

Example:  $\mathcal{B}(\mathcal{H})$ , cts. linear operators on  $\mathcal{H}$

- algebraic structure:  $*$ -algebra,  $\langle T^* \xi, \eta \rangle = \langle \xi, T \eta \rangle$
- analytic structure:  $\|T\| = \sup \{ \|T\xi\|_{\mathcal{H}} ; \|\xi\| \leq 1 \}$ , Banach space
- e.g.  $\mathcal{H} = \mathbb{C}^n \rightsquigarrow n \times n$  matrices,  $M_n(\mathbb{C})$

## \* $C^*$ -algebras

- $A \subseteq \mathcal{B}(\mathcal{H})$ , closed in  $\|\cdot\|$ -topology
- <sup>unital</sup> commutative case:  $C(X)$   
topological flavor

## von Neumann algebras

- $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ , closed under pt-wise limits
- commutative case:  $L^\infty(X, \mu)$   
measure theoretic flavor

## Ex. group algebras

- $\Gamma$ : <sup>countable</sup> discrete grp.  $\rightsquigarrow$  Hilbert sp.  $l^2(\Gamma)$  (square summable  $f: \Gamma \rightarrow \mathbb{C}$ )
- left regular rep:  $\gamma \mapsto \lambda_\gamma \in \mathcal{B}(l^2(\Gamma))$ ,  $\lambda_\gamma(f) = f(\gamma^{-1} \cdot)$   
 $(\delta_\gamma)$   $\lambda_\gamma \delta_{\gamma'} = \delta_{\gamma\gamma'}$
- $C_\lambda^*(\Gamma) = \|\cdot\|$ -closure of  $*$ -subalg. generated by  $\{\lambda_\gamma : \gamma \in \Gamma\}$   
 $\vee N(\Gamma) =$  similar, using pt.-wise lim's.

Generalizes Fourier transform:  $\Gamma$  abelian  $\Rightarrow C_\lambda^*(\Gamma) = C(\hat{\Gamma})$

E.g.  $C_\lambda^*(\mathbb{Z}) = C(\underline{\mathbb{T}}) \not\cong C(\underline{\mathbb{T}^2}) = C_\lambda^*(\mathbb{Z}^2)$

$\vee N(\mathbb{Z}) = L^\infty(\mathbb{T}) \cong L^\infty(\mathbb{T}^2) = \vee N(\mathbb{Z}^2)$

$\swarrow$  dual of  $\Gamma$

## Ex. Dynamics $\alpha: \Gamma \rightarrow \text{Homeo}(X)$

- action  $\Gamma \curvearrowright X \rightsquigarrow$  induced action  $\Gamma \curvearrowright C(X)$ ,  $\gamma f = f \circ \alpha_\gamma^{-1}$
- combine  $C(X)$ ,  $C_\lambda^*(\Gamma)$  — similar to semi-direct product  $N \rtimes H$  of groups  
 $\rightsquigarrow$  crossed product  $C(X) \rtimes_\lambda \Gamma$
- When  $X/\alpha$  is "reasonable" (e.g.  $\alpha$  free, proper),  
 $C(X/\alpha)$  and  $C(X) \rtimes_\lambda \Gamma$  have the "same" representation theory

### Famous example: irrational rotation algebra $A_\theta$

- $\theta \in \mathbb{R} \setminus \mathbb{Q}$ .  $\mathbb{Z} \curvearrowright \mathbb{T}$  by  $e^{2\pi i t} \mapsto e^{2\pi i(t+\theta)}$
- $A_\theta := C(\mathbb{T}) \rtimes_\lambda \mathbb{Z}$

## 2. Factors, finite-dimensional approximations, amenability.

Factor: a vN alg. with no vN alg. ideals ( $\Leftrightarrow$  has trivial center)

Ex.

- $\bigotimes_{1}^{\infty} M_2(\mathbb{C})$  SOT ← normalized matrix traces
- $vN(\Gamma)$ ,  $\Gamma$  infinite and icc ∞ conjug. classes
- $L^{\infty}(X) \rtimes \Gamma$ ,  $\Gamma \curvearrowright X$ : free, ergodic, probability-measure preserving action

- all of these have a trace:  
cts. linear functional  $\tau$   
with  $\tau(xy) = \tau(yx)$
- $\infty$ -dim'l factors with a trace are called II<sub>1</sub> factors.

## Early classification results.

Def. a vN alg. is hyperfinite if it can be approximated internally by finite  $\oplus M_{n_i}(\mathbb{C})$  dimensional alg's

$UF_n$  dense

Ex.

•  $\overline{\bigotimes_1^{\infty} M_2(\mathbb{C})}^{SOT} \cong \overline{\bigotimes_1^{\infty} M_3(\mathbb{C})}^{SOT} \cong \text{vN}(\Gamma)$ ,  $\Gamma =$  finite permutations on  $\mathbb{N}$ , or any locally finite icc  $\Gamma$

43 Theorem (Murray-von Neumann)  $\exists!$  hyperfinite II<sub>1</sub> factor,  $\mathcal{R}$ .

Glimm (1960): not so for  $C^*$ -alg's!  $\overline{\bigotimes_1^{\infty} M_2(\mathbb{C})}^{||\cdot||} \not\cong \overline{\bigotimes_1^{\infty} M_3(\mathbb{C})}^{||\cdot||}$ .

Classified a  $C^*$ -analog of hyperfinite II<sub>1</sub> factors.

## Amenability

$\Gamma$  (discrete group) is amenable if  $\exists$  finitely additive left-invariant probability measure on its subsets (a "mean").

Ex. • finite grps, abelian grps.

• closed under subgroups, extensions, quotients, direct limits

• non-ex. = free groups  $\mathbb{F}_n$  ( $n \geq 2$ ) — see: Banach-Tarski

Can define analog for  $C^*$ - and  $vN$  alg's. Have:  $\underline{C^*_1 \Gamma}$  amenable  $\Leftrightarrow$   $\underline{vN(\Gamma)}$  amenable  $\Leftrightarrow$   $\underline{\Gamma}$  amenable

Definition is abstract — as opposed to concrete def. of hyperfiniteness.



## Breakthrough result:

Theorem (Connes '76): a vN alg. is hyperfinite iff it is amenable.

(In particular, every amenable II<sub>1</sub> factor is  $\cong \mathcal{R}$ .)

• E.g.:  $L^\infty(\mathbb{T}) \rtimes_{\theta} \mathbb{Z}$  is independent of  $\theta \in \mathbb{R} \setminus \mathbb{Q}$

• Led to further breakthroughs, e.g.: all free ergodic prob. measure preserving actions of infinite amenable grps are orbit equivalent.

• a major ingredient: an amenable II<sub>1</sub> factor  $M$  satisfies  $M \cong M \otimes \mathcal{R}$ .

Mc Duff

## Classifying approx. fin. dim'd $C^*$ -algebras.

- internal approximations by fin. dim'd alg's in norm: AF algebras

Theorem (Elliott '77) AF algebras are classified by their  $K_0$  groups.

Op. alg. K-theory: NC extension of Atiyah-Hirzebruch's top K-theory

Distinction:  $L^\infty([0,1])$  — hyperfinite vNa ;  $C([0,1])$  — not AF

### 3. More recent progress

- generalizations? inspiration from vNa side
- Elliott: classifies large class of AT alg's — lim's of  $\oplus M_n(C(T))$  — conjectures classification holds very generally (simple, amenable, ...)
- positive results for more general types of lim's; includes alg's not obviously of that form — e.g.  $A_\theta$

- also challenges: Rørdam, Toms produced counterexamples; not possible to fix by modifying invariant (reasonably)
- Toms-Winter regularity: study properties that make a  $C^*$ -alg. "well behaved"
- analogy with vNa factors, where natural notions of amenability are equiv.



The Toms-Winter <sup>essentially a thm.</sup> conjecture:

•  $A$ : unital, sep., simple, amenable,  $\neq M_n(\mathbb{C})$

TFAE:

(i)  $A$  has finite nuclear dimension

(ii)  $A$  absorbs the Jiang-Su alg.  $\mathcal{Z}$   
tensorially:  $A \otimes \mathcal{Z} \cong A$

(iii)  $A$  has strict comparison of positive elements

- NC analog of covering dim:  
 $\dim_{\text{nuc}} C(X) = \dim X$
- connections to dim theories for groups, dyn. systems...

- analog of McDuff:  $M \otimes \mathcal{K} \cong M$
- $\infty$ -dim'l analog of  $\mathbb{C}$ ; K-theory can't tell them apart

"regular": satisfies one (all) of these conditions

## The Classification Theorem.

Along with J. Gabe, C. Schafhauser, A. Tikuisis, and S. White,  
we have obtained a proof of the following:

Simple, separable, amenable, regular  $C^*$ -alg's in the VCT  
class are classified by K-theory and traces.

Ex. Thm. applies to  $C(X) \rtimes_{\lambda} \Gamma$  if e.g.

- $X$  is compact metric space of finite covering dimension;
- $\Gamma$  is locally of subexponential growth;
- action is free & minimal.

More concrete ex:  $\mathbb{Z} \curvearrowright \mathbb{Z}_2 = \varprojlim \mathbb{Z}/2^n\mathbb{Z} \subseteq \prod \mathbb{Z}/2^n\mathbb{Z}$   
 $\quad \quad \quad \searrow \mathbb{Z}[\frac{1}{2}], \mathbb{Z}$